MOTION OF A SMALL SPHERE IN A NONHOMOGENEOUS FLOW OF AN INCOMPRESSIBLE LIQUID

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We consider the motion of a small sphere in an arbitrary potential flow of an ideal liquid. For the general case we obtain an integral of the equations of motion and a particular solution. We find flows in which the force acting on the sphere is central. We also obtain exact solutions of the equations of motion of the sphere for the cases of stationary flows around a cylinder and around a body of revolution when the forces are noncentral. N. E. Zhukovskii [1] calculated the force acting on a fixed sphere in an arbitrary nonstationary flow. Kelvin [2] obtained the equations of motion of a sphere in a stationary flow of a liquid circulating through a hole in a solid. A formula for the force **F**, acting on a fixed small body of volume V in a stationary flow with speed v, was obtained by Taylor [3]:

$$\mathbf{F} = (\partial T_0 / \partial \mathbf{v})\nabla \mathbf{v} + {}^1/_2\rho V \nabla v^2$$

Here T_0 is the kinetic energy of an unbounded liquid in which a body moves with velocity **v**. This problem was solved in [3] through a direct integration of the pressure forces over the surface of the body in a flow defined by multipoles of the first and second orders at infinity.

1. Basic Equations. In [4, 5] a derivation is given of the equations of motion of a gas bubble and of a body in an arbitrary potential flow of an ideal incompressible liquid on the basis of Lagrange's equations; in these references expressions are obtained for the additive component of the Lagrange function corresponding to the motion of the liquid.

The Lagrange function L for the sphere–liquid system includes, besides the difference of the kinetic energy in the relative motion and the product of the volume of the body by the pressure in the flow [4, 5], also the kinetic energy of the sphere of radius R; thus we have

$$L = {}^1/_3\,\pi R^3\,[\rho\,(\mathbf{q}^\cdot - \mathbf{v})^2 - 4p_0] + {}^2/_3\pi R^3\rho'\mathbf{q}^{\cdot 2}$$

Here ρ and ρ' are, respectively, the densities of the liquid and the sphere; v and p_0 are, respectively, the speed and pressure of the undisturbed flow at the sphere center whose coordinates are given by **q**. In the case of a solid sphere the expression for L can be simplified by adding in the total derivative with respect to the time

$$\frac{d}{dt}\left(\frac{2\pi}{3}R^3\Phi_0\right) = \frac{2\pi}{3}R^3\left(\frac{\partial\Phi_0}{\partial t} + \mathbf{q}^\cdot\mathbf{v}\right)$$

and taking into account the Cauchy–Lagrange integral for the undisturbed flow with potential Φ_0; thus

$$L = {}^1/_3\pi R^3\,[(\rho + 2\rho')\,\mathbf{q}^{\cdot 2} - 6p_0]$$

The function L yields the following equations of motion:

$$(\rho + 2\rho')\,q_\alpha^{\cdot\cdot} = -3\partial p_0/\partial q_\alpha \quad (\alpha = 1,\ 2,\ 3) \tag{1.1}$$

These equations can also be obtained by starting from the expression for the force acting on the sphere moving in a nonhomogeneous flow [6].

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In a stationary flow of an ideal liquid the equations (1.1) assume the form

$$mq_\alpha^{\cdot\cdot} = {}^1/_2\, \partial v^2/\partial q_\alpha \quad (m = (\rho + 2\rho')/3\rho) \tag{1.2}$$

Equations (1.2) correspond to the motion of a particle with mass m in a force field with potential $^1/_2\, v^2$. It follows from Eqs. (1.1) that the acceleration of the sphere is proportional to the acceleration of the liquid. For $\rho' = 0$ (a bubble) the sphere acceleration is equal to three times the acceleration of the liquid. When $\rho' = \rho$ the accelerations of the body and the liquid are the same.

Equation (1.2) has the following particular solution:

$$q_\alpha^{\cdot} = v_\alpha / \sqrt{m}$$

which may be verified by a substitution. This solution shows that for some initial conditions the particles can move along stream lines of the liquid.

The equations of motion (1.2) for a small sphere have a first integral analogous to the energy integral in the dynamics of a particle:

$$mq^{\cdot 2} - v^2 = \text{const} \tag{1.3}$$

For the two-dimensional problem we can find an exact solution if we know yet another first integral. The case of central forces is particularly simple.

2. Flows in Which the Force Is a Central Force. We can prove the following theorem for two-dimensional flow with the complex potential W.

Theorem. The force acting on a small sphere in a two-dimensional flow is a central force if and only if the derivative of the complex potential has the form $dW/dz = cz^\lambda$, where c is an arbitrary complex number and λ is real.

Proof. It follows from Eq. (1.2) that the force is central if

$$dW / dz\,| = f\,(\,|z|\,)$$

where f is a function of a real argument.

Consequently, the real part of the analytic function Ln (dW/dz) is a harmonic function depending only on $|z|$. The general form of such a harmonic function is $\lambda \ln |z| + c$. Hence

$$dW / dz = cz^\lambda$$

As an example of motion with a central force we cite the case of motion at a vortex source whose complex potential is equal to c Ln z. Flows interior to a corner ($W = cz^\lambda$, $\lambda > 1/2$) belong to this class.

For $\lambda = 2$ we obtain a flow about a cylindrical body in the neighborhood of a critical point in the flow.

In the three-dimensional case motion in a source field corresponds to central forces.

In all these cases it is not difficult to write down the exact solutions of the equations (1.2), at least in terms of quadratures, by using the momentum integral instead of the integral (1.3). The integration of Eqs. (1.2) becomes more complicated if the forces are noncentral.

3. Some Exact Solutions in the Case of Noncentral Forces. We consider a small sphere situated in the flow past a circular cylinder. Let the speed of the flow at infinity be equal to one and let the cylinder radius also be equal to one; let r be the distance to the axis of the cylinder and let θ be the angle reckoned from the direction of the flow velocity at infinity. The liquid velocity components are then given by

$$v_r = \cos\theta\,(1 - r^{-2}), \quad v_\theta = -\sin\theta\,(1 + r^{-2})$$

$$v^2 = 1 + r^{-4} - 2\,r^{-2}\cos 2\theta$$

The equation corresponding to Eq. (1.2) for the change in the moment of momentum is

$$\frac{d}{dt}(mr^2\theta^{\cdot}) = \frac{2\sin 2\theta}{r^2}$$

Changing over to a differentiation with respect to θ in the last equation, we can find $r^2\theta^{\cdot}$ as a function of θ.

Thus, Eq. (1.2), for the case of a sphere in the flow around a cylinder, has the the two first integrals

$$m(r'^2 + r^2\theta'^2) - r^{-4} + 2r^{-2}\cos 2\theta = E$$
$$mr^4\theta'^2 + 2\cos 2\theta = H$$

The first equation coincides with Eq. (1.3). The resulting system of equations is equivalent to the following:

$$mr^4r'^2 = 1 - Hr^2 + Er^4$$
$$mr^4\theta'^2 = H - 2\cos 2\theta \tag{3.1}$$

The solution of Eq. (3.1) is expressed in terms of elliptic integrals.

Let the particle move from infinity along with the flow; then $E = m$, $H = mb^2 + 2$ (b is the distance to the line, parallel to the flow velocity and passing through the center, when the sphere is situated at infinity). Putting $\dot{r} = 0$ in the first part of Eq. (3.1), we can find the least distance r_* that the sphere can approach the cylinder axis for a given b:

$$2mr_*^2 = mb^2 + 2 + \sqrt{(mb^2+2)^2 - 4m} \tag{3.2}$$

When $m < 1$ ($\rho' < \rho$) the body does not reach the cylinder surface since $r_*^2 > 1$. It is possible for the particles to come into contact with the cylinder if $m > 1$. Thus the possibility of the small spherical particle being tangent to the cylinder is determined by the ratio of the particle density ρ' to the liquid density ρ. Particles which are denser than the liquid may touch the cylinder; if they are less dense, they do not touch the cylinder.

The expression (3.2) enables us to calculate the capture cross section σ for $m > 1$ ($\rho' > \rho$):

$$\sigma = \pi b_0^2 = \pi(m-1)/m$$

Here b_0 is the largest limiting distance b for which the sphere is tangent to the cylinder surface. From Eq. (3.1) it follows that the point $\theta = \theta_0$ on the cylinder at which tangency occurs for $b = b_0$ is determined from the equation

$$\int_{\theta_0}^{\pi} \frac{d\theta}{\sqrt{1+m-2\cos 2\theta}} = \frac{1}{\sqrt{m}} K\left(\frac{1}{\sqrt{m}}\right)$$

Here K is the complete elliptic integral of the first kind.

In [7] numerical methods were used to obtain the trajectories of a bubble in the flow of a liquid past a circular cylinder. The impossibility of precipitating the bubble onto the cylinder in the flow of an ideal liquid was established. This is in agreement with our result obtained above.

Suppose that the sphere moves in an axially symmetric flow, consisting of a uniform flow and a source

$$v_r = \cos\theta + r^{-2}, \quad v_\theta = -\sin\theta, \quad v^2 = 1 + r^{-4} + 2r^{-2}\cos\theta$$

This flow can be regarded as the flow over an unbounded body whose contour is given by the equation

$$r = 1/\sin(\theta/2) \tag{3.3}$$

In this case the first integrals of equations (1.5) are of the following form:

$$mr^4r'^2 = 1 - Hr^2 + Er^4$$
$$mr^4\theta'^2 = H + 2\cos\theta \tag{3.4}$$

The trajectories of the motion, obtained from equations (3.4), can be written in terms of elliptic integrals. For spherical bodies, moving from infinity along with the flow of the liquid, $E = m$, $H = mb^2 + 2$. For r_* we obtain the same expression (3.2) from the first of the equations (3.4) that we did in the case of cylinder. From the relations (3.2) and (3.3) we obtain an estimate for the smallest distance r from the source for which it is possible to have tangency of the sphere and the body of revolution with Eq. (3.3) as generator. In particular, for a spherical bubble $r > 3 + \sqrt{6} \approx 5.45$.

If we reverse the liquid flow in question, we have the problem of drawing the particles into the flow. In this case equations (3.4) assume the following form:

$$mr^4r'^2 = 1 - (mb^2 - 2)r^2 + mr^4$$
$$mr^4\theta'^2 = mb^2 - 2 + 2\cos\theta \qquad (3.5)$$

We obtain an expression for r_* from the first part of Eq. (3.5):

$$2mr_*^2 = mb^2 - 2 + \sqrt{(mb^2-2)^2 - 4m} \qquad (3.6)$$

Generally speaking, there exists yet another value of r_*, corresponding to the second smaller root of the quadratic equation. However in integrating Eq. (3.5) from $r = \infty$ this second value is not attainable.

From the relation (3.6) it follows that the reverse flow point r_* exists providing that

$$b^2 > (2 + 2\sqrt{m}) / m \qquad (3.7)$$

If the limiting distance satisfies the condition (3.7), then a globule, having attained the minimum distance (3.6), will go off to infinity. But if the condition (3.7) is not satisfied, the globule will then be drawn into the stream.

Thus, the inequality (3.7) determines the smallest limiting distance b_0 for which the spherical particles are not drawn into the flow. The distance r_0 to which a particle approaches the stream is determined from Eq. (3.6).

$$b_0^2 = (2 + 2\sqrt{m}) / m, \quad r_0 = 1/\sqrt{m} \qquad (3.8)$$

For $b = b_0$ the first of Eq. (3.5) integrates into elementary functions. Using the relations (3.8), we can calculate the number of particles N falling per unit time into a flow of capacity Q if the number n of particles per unit volume is known; thus

$$N = \pi n Q (2 + 2\sqrt{m}) / m$$

It is evident from this relation that the quantity N does not depend on the speed of the uniform flow.

LITERATURE CITED

1. N. E. Zhukovskii, "Generalization of a problem of Bjerkness concerning the hydrodynamic forces acting on pulsating or oscillating bodies inside a liquid mass," in: Collected Works, Vol. 2, Hydrodynamics [in Russian], Gostekhteoretizdat, Moscow–Leningrad (1949).
2. W. Kelvin, "On the motion of rigid solids in a liquid circulating irrotationally through perforations in them or a fixed solid," Philos. Mag., 45, (1873).
3. G. I. Taylor, "The forces on a body placed in a curved or converging stream of fluid," Proc. Roy. Soc., Ser. A, 120, No. 785 (1928).
4. V. V. Voinov, O. V. Voinov, and A. G. Petrov, "Hydrodynamic interaction of bodies in an ideal incompressible liquid and their motion in nonhomogeneous flows," Prikl. Matem. i Mekhan., 37, No. 4 (1973).
5. O. V. Voinov and A. G. Petrov, "The Lagrange function for a gas bubble in a nonhomogeneous flow," Dokl. Akad. Nauk SSSR, 210, No. 5 (1973).
6. O. V. Voinov, "On the force acting on a sphere in a nonhomogeneous flow of an ideal incompressible liquid," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 4 (1973).
7. V. I. Il'ichev, A. A. Kanzeba, G. N. Kuznetsov, and A. T. Listrov, "Motion of a gas bubble in the hydrodynamic flow field over a body," Trudy Akust. Inst., No. 6 (1969).